A General Theory on Space and Re-Entry Similar Trajectories

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The two general equations for the lifting re-entry trajectories are presented. The importance of the method that the paper deals with is outlined in that it allows a study of both phases of a return from space, i.e., any re-entry problem, the limiting conic, and the re-entry phase itself, in an unique way. Furthermore it is pointed out that the general similarity rule already obtained for the re-entry equations can be applied to the limiting conic and to other interesting quantities. Some approximate solutions are also presented which are particularly useful in preliminary design problems. Typical results are also given. Finally, comparison is made, with other known methods of solution, from several standpoints.

Nomenclature

C_D = drag coefficient
g = gravity acceleration
h = gravitational constant
m = body mass
n = deceleration coefficient
p = pressure
q = heat rate
r = radius from earth's center
$\bar{u}$ = Chapman's independent variable
w = areal velocity
z = altitude on sea level
k = ballistic coefficient
A = main cross section
B = constant defining laminar heat flow (= 17,600 Btu/ft$^{3/2}$, sec)
C_p = denoted nondimensional value of p property
D = drag
E = total energy
L = lift
Q = total heat
$\mathcal{R}$ = blunt-body nose radius
R = planetary radius
U = Eq. (27)
V = velocity
Z = Chapman's function[8]
α = atmospheric density constant
β = α/R
ϵ = nondimensional limiting total energy
ζ = nondimensional density
η = drag modulation law
θ = flight path angle
χ = nondimensional limiting areal velocity
λ = parameter defining lift modulation
ξ = nondimensional altitude [Eq. (18)]
ρ = density
φ = lift modulation law

Subscripts

$*$ = quantities at deceleration peak
∞ = quantities when density is vanishing
0 = quantities at sea level
1 = reference quantities

Introduction

The return flight path of a space vehicle consists of two separate phases: 1) an approach phase that takes place on a keplerian conic in the wide space, and 2) a re-entry phase that takes place in the atmosphere, and to which the well-known problems of deceleration and heating are connected.

The whole flight path, hereafter referred to as "aerospace trajectory," for a given lift and drag modulation is a function of initial conditions (specified at some reference point) as well as of the body characteristics. In particular, it is possible to find a point (hereafter referred to as *DP*) at distance r_* from the planet center, where the deceleration reaches a peak value. It is possible however to eliminate from the equations the ballistic parameter by writing them in a dimensionless form. Referring the aerospace trajectory to *DP*, taken as origin, and selecting r_* as unit length, a curve is obtained which is referred to as "relative aerospace trajectory." A first similarity law is thus deduced: if two (or more) bodies of different characteristics, at the same point of their relative aerospace trajectory, have the same value of two of the three following quantities: 1) flight path angle, 2) ratio of velocity to local escape velocity $V/(2gr)^{1/2}$, and 3) ratio of acceleration to gravity a/g, then: 1) the relative aerospace trajectories are the same; and 2) the bodies have, at any point of their common aerospace trajectories, equal values of all nondimensional quantities (and, in particular, velocities and accelerations).

The question naturally arises as to how to determine the scale of the various quantities. As far as the distance are concerned, r_* has to be known, and is shown to depend on the body characteristics in quite a simple way. This, of course, is equivalent to saying that, as said, the relative aerospace trajectories are the same for all the bodies, whereas the absolute trajectories are similar according to the ratio of the r_*. It is derived hence that the equality of entry conditions must be satisfied at altitudes that again are in the same similarity ratio.

Another interesting point is where, and what kind of initial conditions must be specified. Two different approaches are possible; the most obvious would be that of specifying the conic along which the space phase takes place (and so, the two parameters of the conic), but it is also very important to specify the maximum tolerable deceleration (which cannot be safely overcome), and the angle at *DP* (on which the scale is shown to be dependnt). This second approach was chosen for this work.

In conclusion, for a given law of lift and drag modulation and for any pair of maximum deceleration and angle at *DP*, charts can be prepared providing 1) parameters of the limiting, or 2) approach conic, 3) relative heat rate peak, and 4) relative total heat. In particular cases, numerical work can also be facilitated by the use of approximate solutions, which are given in closed form in the paper.

Another important point is the relationship between the method here presented, which is exact, and the solutions given by other authors. Analytical, theoretical, and numerical comparison prove the advantages of the present method.

Received by ARS January 17, 1962; revision received May 27, 1964.

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I. General Theory

1. General Assumptions

1) A nonrotating, spherically symmetric planet is considered.

2) The atmosphere surrounding the planet is spherically symmetric too, and is at rest. Its density ρ is varying with height z according to a law that will be later specified.

3) The spacecraft re-entering the atmosphere is considered as a point of mass m; its motion about the center of gravity is assumed to have no effect on the motion of the center of gravity itself.

4) The aerodynamic drag D is expressed by the equation

$$D = \tfrac{1}{2}C_D\rho AV^2 \tag{1}$$

where both the drag coefficient C_D and the main cross section can vary along the trajectory (drag modulation), and V is the speed.

5) The lifting force, either aerodynamic or not, is expressed by the equation $L = D\lambda\varphi$ where λ is a constant parameter, and φ is varying along the trajectory (lift modulation).

2. General Equations

The general equations of the problem are written in this paper by resolving all forces and accelerations along the normal to the trajectory and the local horizontal. Such equations read (see also Ref. 1)

$$\left.\begin{aligned} r^2\frac{d}{dr}\left(\frac{1}{r^2\cos^2\theta}\right) + \frac{h}{w^2} - \frac{2L}{mV^2}\frac{1}{\cos^3\theta} = 0 \\ \frac{d\log(h/w^2)}{dr} + \frac{2D}{mV^2\sin\theta}\left(1 - \frac{L}{D}\tan\theta\right) = 0 \end{aligned}\right\} \tag{2}$$

Here $h = 2gr^2$ is the gravitational constant, r is the distance from the planet, and θ is the flight angle with the horizontal (Fig. 1); the quantity w is the areal velocity, defined, as well known, by

$$w = Vr\cos\theta \tag{3}$$

By introducing the expressions for L and D, and letting

$$C_DA/m = k_*\eta \tag{4}$$

where k_* is a constant and η is a function defining the drag modulation, the following is obtained:

$$\left.\begin{aligned} r^2\frac{d}{dr}\left(\frac{1}{r^2\cos^2\theta}\right) + \frac{h}{w^2} - \rho\frac{k_*\eta}{\cos^3\theta}\lambda\varphi = 0 \\ \frac{d\log(h/w^2)}{dr} + \rho\frac{k_*\eta}{\sin\theta}(1 - \lambda\varphi\tan\theta) = 0 \end{aligned}\right\} \tag{5}$$

Equations (5) relate the unknowns θ and w to the independent variable r. Inasmuch as the system (5) is of the second order, two initial conditions must be specified. This will be done later.

Equations (5) are written in a form that is particularly adequate for re-entry analysis. In fact, as ρ approaches zero (i.e., when the upper portion of the trajectory is considered), they reduce to

$$\frac{d\log(h/w^2)}{dr} = 0 \qquad r^2\frac{d}{dr}\left(\frac{1}{r^2\cos^2\theta}\right) + \frac{h}{w^2} = 0$$

which can be easily integrated to yield

$$w = \text{const} = w_\infty \tag{6}$$

$$E = V^2 - (h/r) = \text{const} = E_\infty \tag{7}$$

The constancy of the areal velocity w, and of the total energy E, assure that the equations of the two-body problem

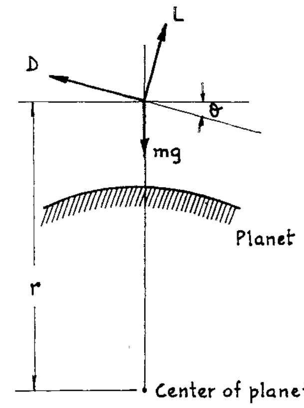


Fig. 1 General definitions.

with air [Eqs. (5)] approach, as $\rho \to 0$, those of the two-body drag-free problem [Eqs. (6) and (7)].

Thus, a continuous linkage is obtained between the interplanetary phase of the space travel and the re-entry phase. Moreover, the limiting values of w and of E can be taken as those specifying the re-entry trajectory: in other words, it is possible to select, as initial conditions,

$$\text{as } \rho \to 0 \qquad \begin{cases} w = w_\infty \\ E = E_\infty \end{cases} \tag{8}$$

However, other points along the trajectory can be selected too, and this will be done in the following sections.

3. Density Law

The equation of equilibrium for an atmosphere at rest under pressure $p(r)$, $dp = -\rho g dr$, can be written as follows:

$$\frac{d\rho}{\rho} = -\frac{gdr}{dp/d\rho} = -\alpha\frac{dr}{r} \tag{9}$$

In Eq. (9), α is the square of the Mach number corresponding to the orbital velocity and is, obviously, a function of r. However, in an isothermal atmosphere, α is not far from being a constant, and so Eq. (9) yields

$$\rho r^\alpha = \text{const} \tag{10}$$

If the quantity α/r, instead of α, is assumed to be a constant, say β, again Eq. (9) yields

$$\rho e^{\beta z} = \text{const} \tag{11}$$

which is the density law commonly accepted in re-entry analysis.

It is thus seen that the difference between Eqs. (10) and (11) is very slight, even if values of r of the order of some hundreds of miles are considered. In other words, the validity of (10) is the same as (11); it will be used throughout this paper, since it lends itself better for similarity purposes. A comparative analysis of Eqs. (10) and (11) is given in Fig. 2.

4. Deceleration Peak

The total deceleration in excess of gravity, hereafter denoted as ng, is given by

$$ng = (L^2 + D^2)^{1/2}/m = \tfrac{1}{2}k_*\eta\rho V^2(1 + \lambda^2\varphi^2)^{1/2} \tag{12}$$

The deceleration reaches a maximum where $d(ng) = 0$, i.e., where

$$\eta V^2 d\rho + \rho V^2 d\eta + \eta\rho 2VdV + \frac{\eta\rho V^2\lambda^2}{1 + \lambda^2\varphi^2}\varphi d\varphi = 0 \tag{13}$$

With the aid of the equation of motion projected on the drag

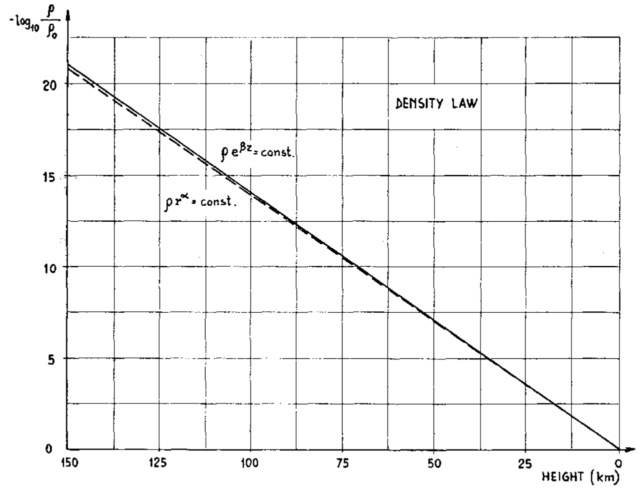


Fig. 2 Density law.

line [a consequence of Eqs. (5)] and of Eq. (10), Eq. (13) yields at DP

$$-\frac{\alpha\eta}{r}+\frac{d\eta}{dr}-\frac{2\eta}{V^2\sin\theta}\left(-\frac{1}{2}k_*\rho\eta V^2+g\sin\theta\right)+\frac{\eta\lambda^2\varphi}{1+\lambda^2\varphi^2}\frac{d\varphi}{dr}=0 \quad (14)$$

The function $k = C_DA/m$ was expressed by $k_*\eta$, and now the function η is set equal to unity at deceleration peak (DP); thus, k_* is the value of k at DP. Furthermore, by remembering the definition of ng [Eq. (12)], the following is obtained:

$$\frac{k_*\rho_*r_*}{\sin\theta_*}=\frac{\alpha-r_*[(d/dr)\log\eta(1+\lambda^2\varphi^2)^{1/2}]}{1-(\sin\theta_*/n_*)(\lambda^2\varphi_*{}^2+1)^{1/2}} \quad (15)$$

where the subscript $_*$ denotes values at DP.†

It seems reasonable that lift and drag modulation be performed with respect to r/r_* or to any other dimensionless quantity. Lift and drag modulation can also include the variation of C_D vs L/D; in this case, of course, additional parameters (defining the aerodynamic behavior of the body) shall enter into the problem. The importance of such parameters has been pointed out in Ref. 2, and, more recently, in Ref. 3, by Professor Ferri. In any case, for a prescribed law of η and λ, φ, letting

$$\alpha_*=\frac{[\alpha-([d/d(r/r_*)]\log\{\eta(1+\lambda^2\varphi^2)^{1/2}\})_*]}{[1-(\sin\theta_*/n_*)(\lambda^2\varphi_*{}^2+1)^{1/2}]^{-1}} \quad (16)$$

Eq. (15) can be written

$$k_*\rho_*r_*/\sin\theta_*=\alpha_* \quad (17)$$

which is the final equation connecting the quantities at DP.

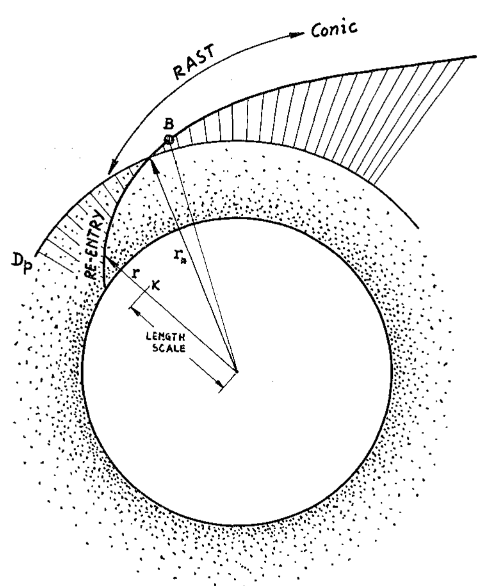


Fig. 3 Similar aerospace trajectories.

† In general, the angle θ_* is very small, whereas the coefficient n_* is of several unities; thus the denominator of the right side of Eq. (15) is very close to 1.

5. Reduction to a Single Equation and Nondimensional Form

By eliminating w in Eqs. (5), and with the nondimensional altitude,

$$\xi=1-(r_*/r) \quad (18)$$

remembering Eqs. (17) and (10), the following is obtained:

$$(1-\xi)^2\frac{d}{d\xi}\log\left\{(1-\xi)^\alpha\frac{n\lambda\varphi}{\cos^3\theta}\alpha_*\sin\theta_*-\frac{d}{d\xi}\times\left[\frac{(1-\xi)^2}{\cos^2\theta}\right]\right\}(\alpha_*\sin\theta_*)(1-\xi)^\alpha\frac{\eta}{\sin\theta}\times(1-\lambda\varphi\tan\theta)=0 \quad (19)$$

In Eq. (19), the only parameters appearing are θ_* and the planetary constant α (since α_* is depending upon α itself, and upon the lift and drag modulation, which are assumed to be prescribed). In other words, the body characteristics, represented by k_*, have been eliminated. Since L and D are modulated on dimensionless quantities with respect to DP, every term in Eq. (19) is dependent only on such dimensionless quantities. And, for a general theorem, similarity will hold for any other quantity with respect to its DP value.

Obviously, if variation of C_D vs L/D is to be included, the corresponding additional aerodynamic parameters are not eliminated. The reason for which Eq. (10) instead of Eq. (11) has been chosen is also clear. Only a homogeneous law in r/r_* allows one to express ρ/ρ_* independently of the body; with Eq. (11), one would have

$$(\rho/\rho_*)e^{-\beta(r-r_*)}=e^{-\beta r_*[\xi/(1-\xi)]}$$

which depends (although slightly) on the body through r_* [Eq. (17)].

The asymptotic behavior of Eq. (19) is obtained as ξ approaches 1, i.e.,

$$\frac{d^2}{d\xi^2}\left[\frac{(1-\xi)^2}{\cos^2\theta}\right]=0$$

From (6) and (7), letting

$$\varkappa=w_\infty{}^2/hr_* \qquad \epsilon=E_\infty/(h/r_*) \quad (20)$$

the nondimensional form of the limiting conic is obtained:

$$\varkappa[(1-\xi)^2/\cos^2\theta]=1-\xi+\epsilon \quad (21)$$

which is defined by the two nondimensional parameters ϵ and $\varkappa$.

The boundary conditions to Eq. (19) are now to be established. Since ϵ and $\varkappa$ define, as seen, the asymptotic behavior of the spacecraft motion, boundary conditions may prescribe the following limiting values:

$$\text{as } \xi\to 1 \qquad \varkappa[(1-\xi)^2/\cos^2\theta]=1-\xi+\epsilon \quad (22)$$

From a numerical standpoint, this seems to require a guess on θ^*; this can be avoided by making calculations on the dimensional equations (with a dummy value of k_*), and by nondimensionalizing with respect to the DP as found.

An alternative procedure, even more interesting from an engineering viewpoint, may be prescribing values at DP. A very important parameter, as already pointed out, is the maximum deceleration, or n_*. And, since

$$n_* g_* = \tfrac{1}{2} k_* \rho_* V_*^2 (1 + \lambda^2 \varphi_*^2)^{1/2}$$

from the first of Eqs. (5), one gets

$$\text{at } \xi = 0 \qquad \frac{d\theta}{d\xi} = \cot\theta_* + \frac{\alpha_*}{2} \times \left(\lambda\varphi_* - \frac{(1 + \lambda^2\varphi_*^2)^{1/2}}{n_*} \cos\theta_* \right) \qquad (23)$$

Equation (23), together with

$$\text{at } \xi = 0 \qquad \theta = \theta_* \qquad (24)$$

provides the initial conditions associated with Eq. (19).

Thus, for a prescribed modulation [i.e., for prescribed values of λ, $\varphi(\xi)$, $\eta(\xi)$], a solution corresponding to a given n_* and to a given θ_* is obtained independently of the body characteristics: such a solution is hereafter referred to as a "similar solution." A step-by-step integration provides a pair of limiting values for ϵ and $\varkappa$, and so the relationships

$$\left.\begin{aligned} \epsilon &= \epsilon(n_*, \theta_*) \\ \varkappa &= \varkappa(n_*, \theta_*) \end{aligned}\right\} \qquad (25)$$

are obtained, relying quantities at DP to the limiting conic.

6. General Conclusions and Similarity Law

The preceding considerations allow us to draw some general conclusions from the viewpoint of similarity law. A body traveling along its flight path is considered, and the radius corresponding to its DP, say r_*. The curve $(r - r_*)/r_*$ is referred to as "relative aerospace trajectory" (RAST) (Fig. 3).

The dimensionless quantities θ, $V^2/2gr$, n, will vary along the RAST; however the following similarity law holds: If two or more bodies of different characteristics and the same prescribed law of $\varphi(\xi)$ and $\eta(\xi)$, at the same point of their respective RAST, have the same value of two of the three quantities θ, namely, $V^2/2gr$, n, then 1) their RASTS are coincident, and 2) the bodies travel on the common RAST with the same values of θ, $V^2/2gR$, n, at the same point of RAST.

The question naturally arises as to how one determines the real aerospace trajectory, for which the value of r_* must be known. This is simply done through the following relationship:

$$k_* \rho_* r_* = \alpha_* \sin\theta_* \qquad (17')$$

To lend perspective to the reasoning, it is possible to represent the RAST by one curve only (Fig. 3); the scale length (i.e., r_*) is given by Eq. (17′) once k_* is given.

If one considers, for instance, two bodies (1 and 2), with $k_1/k_2 = 100$, one obtains $r_{1*} \cong r_{2*} \cong 20$ miles, and for the point, say B, of their common RAST, such that $(r - r_*)/r_*' = 0.01$, one has $r_1 - r_2 = 20.2$ miles.

Another important point is at which altitude, and which kind of initial conditions must be specified. Two approaches are possible: either to specify the limiting conic or the maximum deceleration (of highest engineering and biological interest) and of θ_* (which, as seen, determine r_*). This approach is chosen in this work; for any pair of n_* and θ_*, a relative conic, (defined by ϵ and $\varkappa$) can be determined (of course, for the actual conic, the value of r_* must be specified, as stated previously).

It is also possible to show[4] that several other quantities of engineering quantities, such as areal velocity w, heat $Q(\xi)$ transferred to the body from infinity to a prescribed ξ, heat rate $q(\xi)$, and time t, can be expressed as the product of a dimensional unit quantities (denoted by the suffix 1) by a "similar" function that is only dependent on the values of θ_*, n_*, and $\theta = \theta(\xi)$. Such dimensionless quantities, denoted as C_V, C_w, C_Q, C_q, etc., are obtained through simple algebra in Ref. 4. A complete summary is given in Table 1.

Again it should be pointed out that the foregoing similarity law holds 1) for all dimensionless quantities, and 2) for a prescribed lift and drag modulation, made with respect to any dimensionless quantity.

The similarity law as described previously can be used also for specific problems, such as guidance requirement analysis.[5]

Table 1 Summary of engineering quantities[a]

Quantity	Symbol	Unit values	Similar function
Radius	r	$r_1 = r_*$	$C_r = \dfrac{1}{1-\xi}$
Areal velocity	w	$w_1 = (g_* r_*^3)^{1/2}$	$C_w = \left[\dfrac{1}{U} \dfrac{2n_*\cos^2\theta_*}{\alpha_* \sin\theta_* (1+\lambda^2\varphi^{*2})^{1/2}}\right]^{1/2}$
Velocity	V	$V_1 = (g_* r_*)^{1/2}$	$C_V = \left[\dfrac{2n_* \cos^2\theta_*}{\alpha_* \sin\theta_* (1+\lambda^2\varphi_*^2)^{1/2}}\right]^{1/2} \dfrac{(1-\xi)U^{-1/2}}{\cos\theta}$
Deceleration	ng	g_*	$C_g = \dfrac{n_* \cos^2\theta_*}{(1+\lambda^2\varphi_*^2)^{1/2}} \dfrac{(1-\xi)^{\alpha+2}(1+\gamma\lambda^2\varphi^2)^{1/2}}{U \cos^2\mu}$
Angular range	φ	...	$C = \displaystyle\int_0^\nu \frac{d\xi}{(1-\xi)\tan\theta}$
Time	t	$t_1 = \left(\dfrac{r_*}{g_*}\right)^{1/2}$	$C_t = \left(\dfrac{\alpha_* \sin\theta_*}{2n_* \cos^2\theta_*}\right)^{1/2} (1+\lambda^2\varphi_*^2)^{1/2} \displaystyle\int_0^\xi \frac{U^{1/2} d\xi}{(1-\xi)^3 \tan\theta}$
Heat rate	q	$q_1 = \dfrac{B}{\mathcal{R}^{1/2}} \left(\dfrac{8}{k_*\rho_* r_*}\right)^{1/2} \left(\dfrac{R}{r_*}\right)^{3/2}$	$C_q = \left(\dfrac{n_* \cos^2\theta_*}{(1+\lambda^2\varphi_*^2)^{1/2}}\right)^{3/2} \dfrac{1}{\alpha_* \sin\theta_*} \dfrac{(1-\xi)^{\alpha/2+3}}{U^{3/2}\cos^3\theta}$
Total heat	Q	$Q_1 = q_1 t_1$	$C_Q = \dfrac{n_* \cos^2\theta_*}{(2\alpha_* \sin\theta^*)^{1/2}} \dfrac{1}{(1+\lambda^2\varphi^{*2})^{1/2}} \displaystyle\int_0^\xi \frac{(1-\xi)^{\alpha/2} d\xi}{U \cos^2\theta \sin\theta}$

[a] $\left[U = \dfrac{w_*^2}{w^2} = \exp\left(-\alpha_* \sin\theta_* \displaystyle\int_0^\xi \frac{(1-\xi)^{\alpha-2}}{\sin\theta} \eta[1 - \lambda\varphi \tan\theta]\, d\xi\right)\right]$.

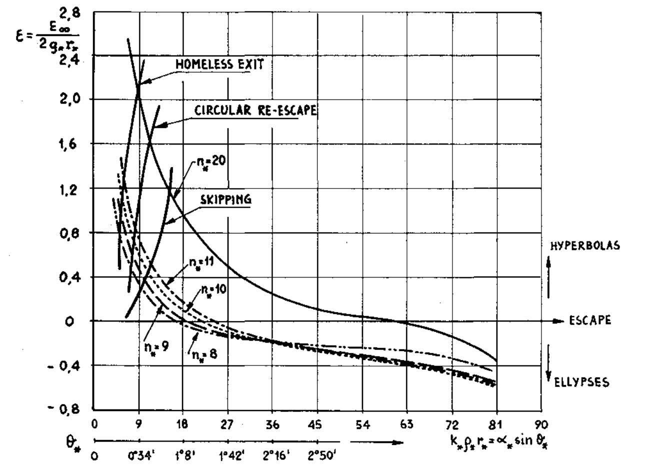


Fig. 4 Variation of total energy of limiting conics [L/D = 0, ($E = V^2 - 2gr$)].

II. Typical Results

1. Limiting Conic Parameters

On the basis of the theory presented previously, a general investigation was undertaken for the case of a constant lift/drag ratio up to DP, and subsequent lift cutoff. This is by no means an optimum lift programing, but it is deemed to give a general idea on the effect of the various parameters involved in the analysis.

Several quantities were investigated. The first of them is the total energy of the limiting conic, $\epsilon = E_\infty/2g_*r_*$, whose values are represented in Fig. 4, for $L/D = 0$. Positive values of ϵ correspond to hyperbolas as limiting conics, and negative values to ellipses, the horizontal axis representing parabolic re-entry conditions.

Velocity at DP is easily seen to have the value $[2n_*g_*r_*/\alpha_* \sin\theta_*(1 + \lambda^2\varphi_*{}^2)^{1/2}]^{1/2}$, and so, for a fixed n_*, it increases with decreasing θ^*, up to a point where centrifugal forces become so high that, little after DP, an unnecessary skipping phase will occur. Again, according to the foregoing expression of V_*, it can be seen that the limiting value of skipping conditions is increasing with increasing n_*.

The corresponding results for $L/D = 0$ are summarized by the curve referred to as "skipping" in the left side of the diagram. Decreasing values of $\sin\theta_*$ beyond skipping limit will produce greater and greater centrifugal forces, and consequently 1) conditions of circular re-escape (described by the curve of the diagram) and 2) conditions of homeless re-exit into space. The two curves seem to approach each other.

They can be referred to as "overshoot boundary" for the various re-entry conditions. Undershoot boundaries are

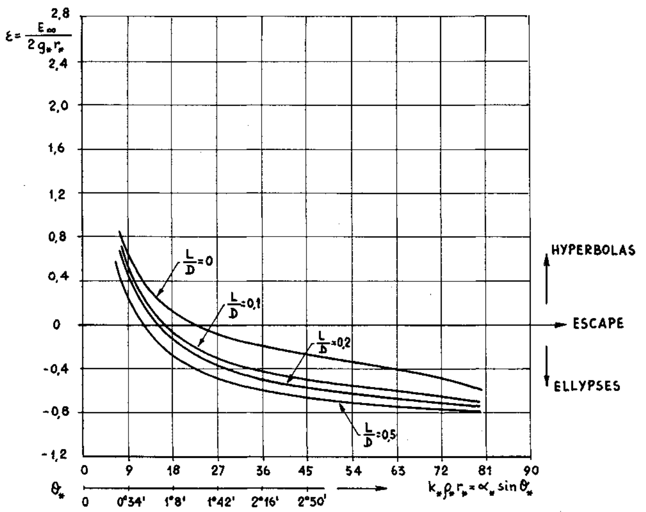


Fig. 5 Variation of total energy of limiting conics [n = 10, ($E = V^2 - 2gr$)].

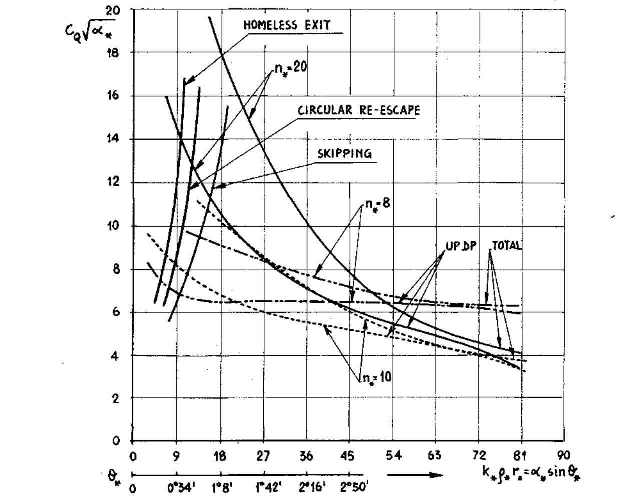


Fig. 6 Variation of dimensionless total heat transferred during re-entry [$L/D = 0 =$ const, ($\alpha_* = \alpha$)].

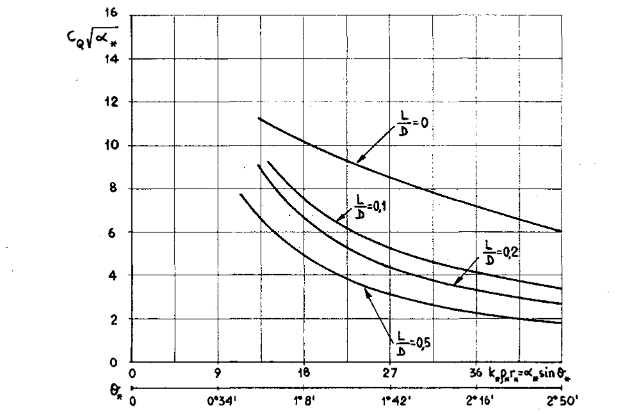


Fig. 7 Variation of dimensionless total heat transferred during re-entry ($n_* = 10$).

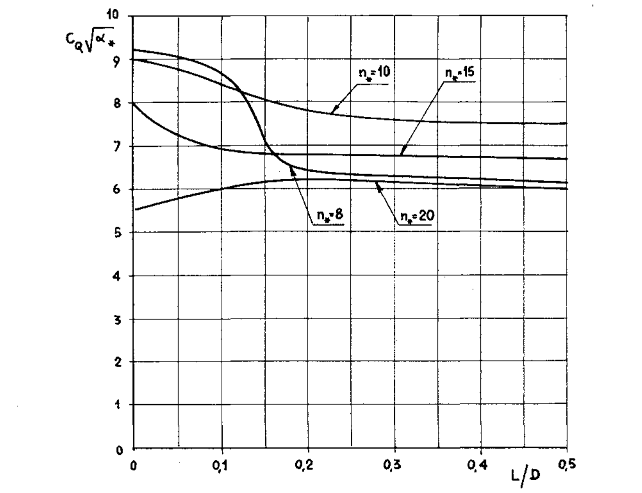


Fig. 8 Variation of total heat vs L/D for parabolic return conditions.

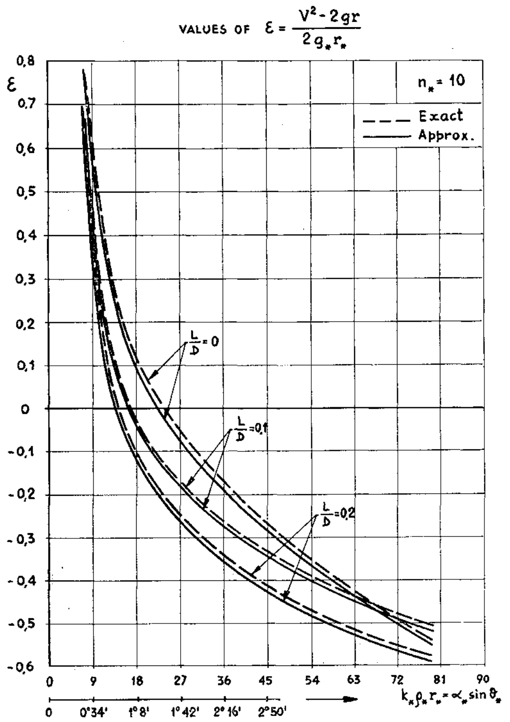


Fig. 9 Comparison approximate to exact solution.

represented by the curves $n_* =$ const which are directly given on the plot.

It is seen that, for lower values of n_*, return from space is restricted to small intervals in θ_*, if skipping phases or dangerous re-exits are to be avoided. Similar calculations were performed for $L/D = 0.1 + 0.2 + 0.5$.[4]

To lend more perspective to the diagram, and to make comparison from the viewpoint of lift/drag ratio, the various curves corresponding to a given n_* and to different L/D can be plotted on a same diagram. For sake of example only, the case $n_* = 10$ was considered, with the results given in Fig. 5 that is self-explanatory.

2. Heat Transferred

Another group of results refer to total heat transferred to the body (Figs. 6 and 7). The curves exhibit a pronounced trend to decrease with increasing θ_*. This is because of the decreasing values of V at DP, which is generally not too far more than the peak of heat rate; another more important factor is because of the circumstance that steeper angles correspond to shorter duration of the heating phase.[1]

Heat is transferred to the body after DP also. Its values should be, however, in connection with the lift programing, which is not necessarily a cutoff at DP. For this reason, two curves are plotted for every n_* and L/D, one corresponding to total heat prior to DP, and the other one including also that after DP.

The effect of the latter is seen to be quite important; this leads to the conclusion that a good lift programing after DP should also be studied so as to minimize total heat.

With the use of the forementioned diagrams, some interesting conclusions can be drawn; for instance, all the parabolic return conditions can be obtained on the plots, observing the intersections at $\epsilon = 0$.

The corresponding pairs of n_* and θ_* provide the values of C_Q. Such values are represented in Fig. 8 for various values of n_*; they can be quoted as an extension and an improvement of similar results of Ref. 1. The beneficial effect of lift[1, 6] is evident from the plot, and is seen to be specially limited to small values of L/D. Higher values would not produce a proportional benefit.

3. Approximate Solution

Several approximate solutions are obtained in Ref. 4. One, among the most simple and handy, is obtained by setting, in the first of Eqs. (5),

$$U = \frac{w_*^2}{w^2} = \sum_0^3 S b_S \left(\frac{\rho}{\rho_*}\right)^S \qquad \cos^3\theta \simeq \cos^3\theta_*$$

and by imposing the values of U, $dU/d\rho$ at DP and at infinity. Simple algebra then provides the values of the coefficients b_S, and subsequently the value of U at infinity.

Such solution was applied to the case $n_* = 10$ for different values of L/D. The corresponding values of ϵ vs θ_* are

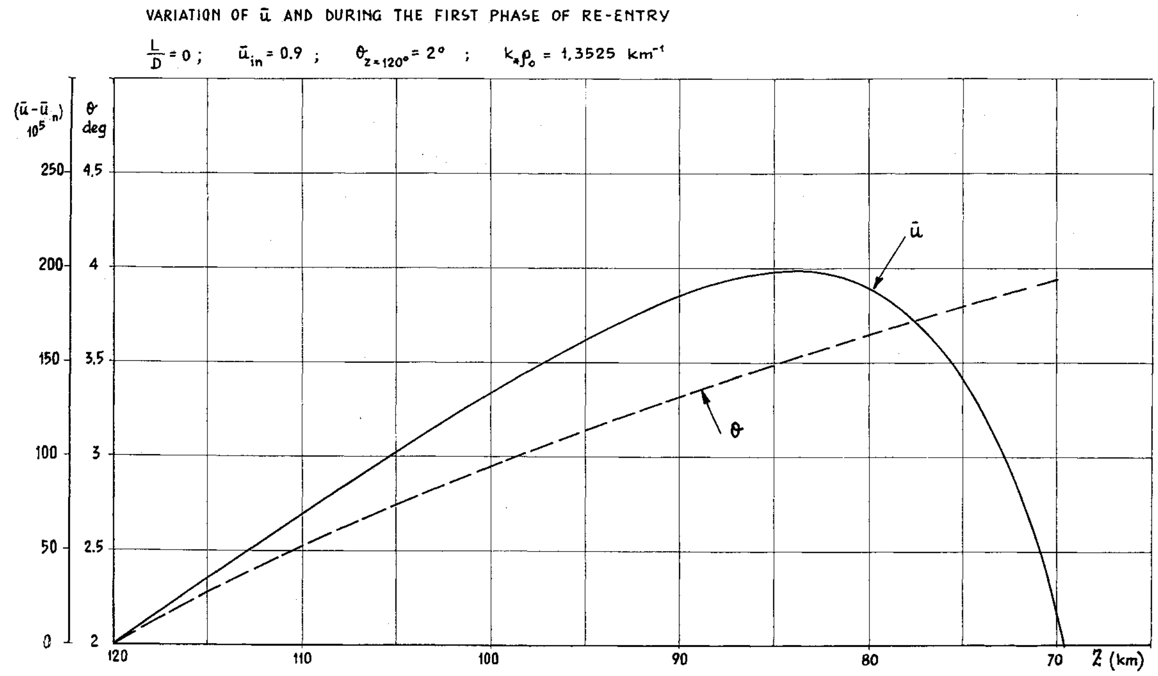


Fig. 10 Evaluation of Chapman's Z function.

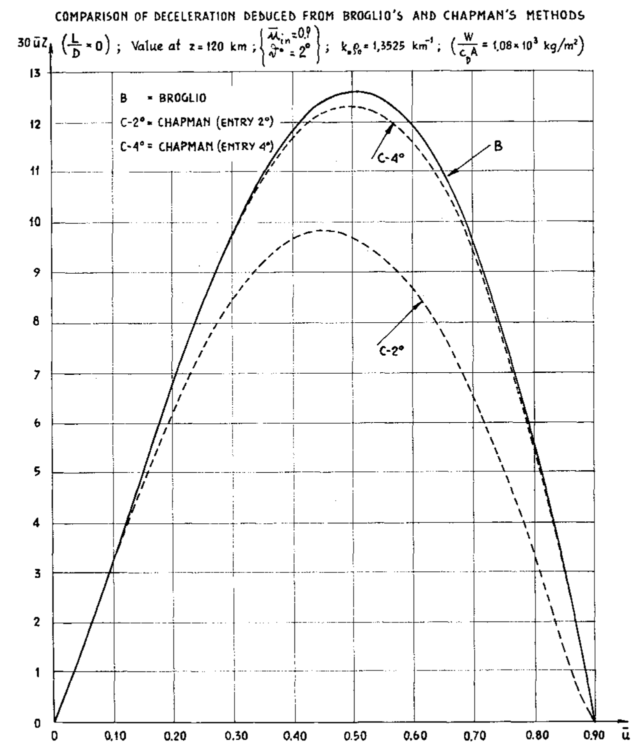
plotted in Fig. 9, where comparison is made with the corresponding exact values.

The very small, unappreciable differences arising in two sets of curves certify the good agreement. Other methods of solution can be found in Refs. 4 and 7.

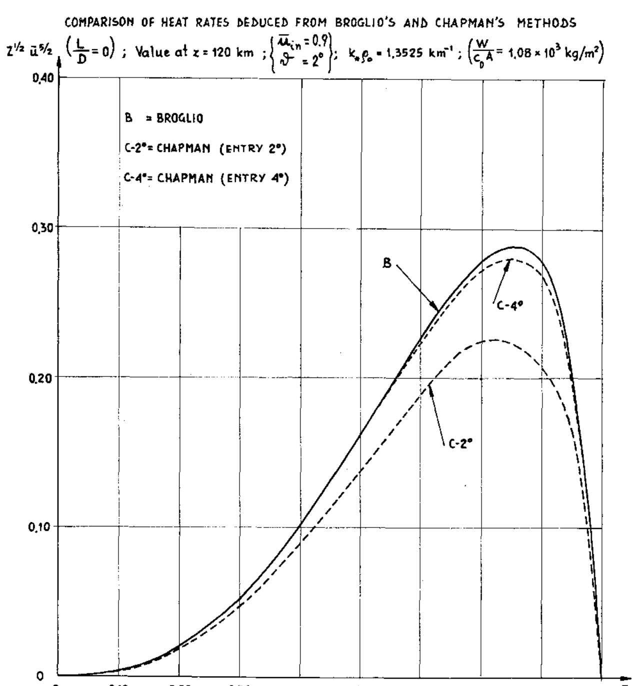
III. Comparison with Other Methods

1. Comparison of Basic Equations

The well-known method of Ref. 8 allows one also to deduce a similarity law, since the ballistic parameter is eliminated; moreover, the influence of the planet is also lacking, since nothing in Chapman's final equation refers to quantities such as h or α. This is obtained, however, by discarding some terms in the basic equations [in accordance with the basic assumption $d(V\cos\varphi)/V\cos\varphi \gg dr/r$]. Also, the term $L/D\tan\varphi$ of the second of Eqs. (1) is neglected with respect to 1.



a)



b)

Fig. 11 Comparison of heat flow deduced from Broglio's and Chapman's method.

Some comments can be made referring to the preceding circumstances.

1) The neglecting of the said terms leads, as recognized by Chapman himself, to heavy errors in the evaluation of the limiting conics, i.e., no linkage between the outer and the re-entry trajectory are possible. With the introduction of Chapman's assumptions, Eqs. (5) become

$$\left.\begin{aligned} r^2\frac{d}{dr}\left(\frac{1}{r^2\cos^2\theta}\right)+\frac{2}{r\cos^2\theta}+\frac{h}{w^2}-\frac{L}{D}\frac{k\rho}{\cos^3\theta}&=0\\ \frac{d\log(h/w^2)}{dr}+\frac{2}{r}+\frac{k\rho}{\sin\theta}&=0\end{aligned}\right\}\quad(26)$$

which yield $V\cos\theta = \text{const}$ instead of the exact limiting property $Vr\cos\theta = \text{const}$; i.e., no limiting conic is attained.

2) The elimination of the parameter α does not seem to be of special value, since very few planets are presently known, and not only are the parameters, but also the type of the law of density vs altitude is not known. However, the present method also, with the discarding of terms like Chapman's, allows one to eliminate α. In fact, if a new variable the nondimensional density is taken

$$\zeta = (r_*/r)^\alpha = \rho/\rho_*$$

elimination of w^2 among Eqs. (26) yields

$$-\alpha\zeta^{1+(1/\alpha)}\frac{d}{d\zeta}\log\left[\frac{L}{D}\frac{\alpha_*\sin\theta_*}{\cos\theta}\zeta+\alpha\zeta^{1+(1/\alpha)}\frac{d}{d\zeta}\tan^2\theta\right]+\frac{\alpha_*\sin\theta_*}{\sin\theta}\zeta+2\zeta^{1/\alpha}=0$$

Obviously, the quantity $1/\alpha$ is negligible as compared to unit in the exponentials. Thus, if $\alpha = \alpha_*$, i.e., if no modulation is made, the foregoing equation simply reduces to

$$-\zeta\frac{d}{d\zeta}\log\left\{\left[\frac{L}{D}\frac{\sin\theta_*}{\cos^3\theta}+\frac{d}{d\zeta}\tan^2\theta\right]\right\}+\frac{\sin\theta_*}{\sin\theta}\zeta=1$$

whence α is eliminated. It should be pointed out, however, that the possibility of elimination essentially is based on neglecting the previously stated terms, which, as previously said, leads to serious errors in the evaluation of the limiting behavior.

2. Analytical Comparison

No difficulty arises, as seen, in the step-by-step integration of Eq. (19). On the contrary, Chapman's equation presents a singularity, pointed out by the author himself at the beginning. For the first step, the solution (in Chapman's notation)

$$Z_0 = (\beta r)^{1/2}\bar{u}_0\left[(\sin\varphi_i)\log\frac{\bar{u}}{\bar{u}_i}-\frac{\cos^3\varphi_i}{2}\left(\frac{L}{D}\right)\ln^2\frac{\bar{u}}{\bar{u}_i}\right]$$

$$Z_0' = (\beta r)^{1/2}\sin\varphi_i+\frac{Z_0}{\bar{u}_0}$$

is suggested in Ref. 8.

By letting $\bar{u} = \bar{u}_i + \epsilon$, since ϵ is small, it is possible to write (for example, for $L/D = 0$)

$$Z_0 \simeq (\beta r)^{1/2}\epsilon\sin\varphi_i$$

For the second step, the value of Z'' must be obtained from the differential equation

$$\bar{u}Z''-Z'+\frac{Z}{\bar{u}}=\frac{1-\bar{u}^2}{\bar{u}Z}\cos^4\varphi$$

and it is clearly seen that such Z'' almost entirely depends on the value of Z, i.e., of ϵ, i.e., of the length of the first step. Since this is obviously impossible, it is not clear how the calculations of Ref. 8 have been performed.

3. Numerical Comparison

For the previously stated reasons, it can be expected that, in some cases, and for some purposes, Chapman's method may be not fully adequate. This is by no means a criticism to Chapman's very valuable work, since he also recognized the limitations of his approximate solution and provided a test for the upper bound of the initial altitude. Such a bound is indicated by the author as the height at which, for any specific case under concern, the value of dr/r is of the order of 0.1 $d\bar{u}/\bar{u}$. When such limitation is satisfied, the accuracy is very good, as is demonstrated in many numerical examples of Ref. 2.

In other cases, however, the possibility arises of multiple values of $\bar{u}$ during re-entry. This situation occurs for very low initial aerodynamic forces with respect to the gravity acceleration; in such cases, $\bar{u}$ shows an increasing trend at the beginning (by the reason that the acceleration of gravity is prevailing on the aerodynamic forces). Increasing velocity means also increasing aerodynamic forces, so that $\bar{u}$ reaches a maximum and a subsequent decrease, attaining again the initial values.

This situation is well depicted by the following numerical example: a re-entry with $v^2/gr = 0.81$; $\theta_0 = 2°$; $k_*\rho_0 = 1.3525$ km^{-1} (where ρ_0 is the sea-level density), corresponding to a W/C_DA of 1.08 10^3 kg/m^2. Figure 10 shows the value of $\bar{u}$ and of the angle θ vs the height Z (initial height = 120 km), where the multiple value of $\bar{u}$ is clearly indicated. It seems that the real re-entry condition is 4° instead 2°. The following conclusion is better substantiated by Figs. 11a and 11b. In Fig. 11a, the curve labeled C-2° is one deduced by Chapman's work (Fig. 5, p. 66 of Ref. 2) for 2°, the curve labeled C-4° is deduced from the same source for 4°. The curve labeled B is the one calculated through the exact solution presented in this work. A similar remark applies to Fig. 11b.

In conclusion, it can be stated that the similarity law proposed by Chapman can, whenever the limitations stated by him do not apply, be replaced by that proposed in this paper.

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